

iso-spectral Euler-Bernoulli beams à la Sophus Lie

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Abstract

We obtain iso-spectral Euler-Bernoulli beams by using factorization and Lie symmetry techniques. The canonical Euler-Bernoulli beam operator is factorized as the product of a second-order linear differential operator and its adjoint. The factors are then reversed to obtain iso-spectral beams. The factorization is possible provided the coefficients of the factors satisfy a system of non-linear ordinary differential equations. The uncoupling of this system yields a single non-linear third-order ordinary differential equation. This ordinary differential equation, referred to as the *principal equation*, is analyzed and solved using Lie group methods. We show that the principal equation may admit a one-dimensional or three-dimensional symmetry Lie algebra. When the principal system admits a unique symmetry, the best we can do is to depress its order by one. We obtain a one-parameter family of solutions in this case. The maximally symmetric case is shown to be isomorphic to a Chazy equation which is solved in closed form to derive the general solution of the principal equation.

1 Introduction

A recent study [1] suggests that efforts to model the transverse motion of vibrating beams date back to Leonardo da Vinci. In his discussion of the bending of beam/spring with rectangular cross-section he wrote [2]: “Of bending of the springs: if a straight spring is bent, it is necessary that its convex part become thinner and its concave part, thicker. This modification is pyramidal, and consequently, there will never be a change in the middle of the spring”. About

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¹ To the loving memory of my brother Léopold Fotso Simo.

hundred years after Da Vinci's attempt to develop a beam theory, Galileo suggested an erroneous calculation of the load carrying capacity of a transversely loaded beam. Da Vinci did not benefit from Hooke's law and calculus which postdate him whereas Galileo made the incorrect assumption that under transverse loading, the stress is uniformly distributed on cross-sections. The first correct and systematic formulation of an elasticity theory was done by Jacob Bernoulli (1700-1782). He found that the curvature of an elastic beam at a given point is proportional to the bending moment at that same point. Taking advantage of his uncle seminal work, Daniel Bernoulli(1707-1783) derived the partial differential equation that governs the motion of a vibrating beam. Leonard Euler(1707-1705) pursued the study of the Benoullis by using and extending their theory in his investigation of the shape of elastic beams subjected to various external forces. For a recent review of beam theories, the reader is referred to the paper by Han, Benaroya and Wei [3] and references therein.

In Euler-Bernoulli beam theory, the transverse motion of a thin non-uniform beam is governed by the partial differential equation

$$\frac{\partial^2}{\partial x^2} \left[E(x) I(x) \frac{\partial^2 u}{\partial x^2} \right] + \rho(x) \alpha(x) \frac{\partial^2 u}{\partial t^2} = 0, \quad 0 \leq x \leq L, \quad (1)$$

where E is the modulus of elasticity (a.k.a Young's modulus), $\alpha(x)$ is the cross-sectional area at x , $I(x)$ is the area moment of inertia of the cross-section located at x about a neutral axis, ρ is the density, and L is the length of the beam. Equation (1) is solved subject to appropriate boundary conditions at $x = 0$ and $x = L$. Some common boundary conditions and their physical meaning are

$$\frac{\partial^2 u}{\partial x^2} = 0, \quad u = 0 \quad (\text{hinged end}), \quad (2)$$

$$\frac{\partial u}{\partial x} = 0, \quad u = 0 \quad (\text{clamped end}), \quad (3)$$

$$\frac{\partial^2 u}{\partial x^2} = 0, \quad \frac{\partial}{\partial x} \left(E(x) I(x) \frac{\partial^2 u}{\partial x^2} \right) = 0 \quad (\text{free end}), \quad (4)$$

$$\frac{\partial u}{\partial x} = 0, \quad \frac{\partial}{\partial x} \left(E(x) I(x) \frac{\partial^2 u}{\partial x^2} \right) = 0 \quad (\text{sliding end}). \quad (5)$$

The ansatz $u(t, x) = e^{i\omega t} Y(x)$, where ω is the frequency of vibrations, solves Eq. (1) provided

$$\frac{d^2}{dx^2} \left(f(x) \frac{d^2 Y}{dx^2} \right) = \omega^2 m(x) Y, \quad 0 \leq x \leq L, \quad (6)$$

where

$$f(x) = E(x) I(x), \quad m(x) = \rho(x) \alpha(x). \quad (7)$$

Under Barcilon's transformation [4,5]

$$z = \int_0^x (m/f)^{1/4}, \quad Y = (m^3 f)^{-1/8} U, \quad (8)$$

Equation (8) becomes

$$\frac{d^4 U}{dz^4} + \frac{d}{dz} \left[A(z) \frac{dU}{dz} \right] + B(z)U = \omega^2 U, \quad 0 \leq z \leq l, \quad (9)$$

where the correct expression of A and B are given by Gottlieb [5]. Save for the clamped end conditions, Barcilon's transformation leaves the other boundary conditions invariant under some constraints given explicitly in [5]. As we shall see in the sequel, the canonical form (9) plays a crucial role in inverse spectral problems associated with Euler-Bernoulli beams.

In many practical problems such as the structural identification through non-destructive means, one is interested in determining the physical characteristics of the beam i.e. $f(x)$ and $m(x)$ from few frequencies (spectra). That is, one tries to 'hear' the shape of the beam. Since the vibration of uniform beams (i.e. f and m are constant functions) is well understood, an interesting problem related to the inverse spectral problem is to find non-uniform beams that have the same length and share the same spectra with the uniform beam under the same boundary conditions. This problem was comprehensively studied by Gottlieb [5] using the canonical form (9). He found seven classes of clamped beams that are iso-spectral with the standard unit-coefficient beam ($A = 0 = B$). Some of these beams were rediscovered by Abrate [6] (see [7] for comments) using a different paradigm. For discrete beams, Gladwell [8], introduced two procedures for finding iso-spectral families. His first procedure relies on shifted orthogonal matrix triangulation whereas the second utilizes Toda flow.

This paper is dedicated to the search of iso-spectral beams using Lie symmetry methods together with a factorization method initiated by Pöschel and Trubowitz [9] for Sturm-Liouville operator. To be more precise, the factorization method for the canonical beam operator leads to a system of nonlinear ordinary differential equation that we shall solve by Lie's reduction method. We have organized the present paper as follows. There are four sections of which this introduction is the first. In section 2, we briefly motivate the use of factorization in the search of iso-spectral beams. The factorization approach leads to a system of nonlinear ordinary differential equations. We employ Lie's symmetry theory to study its continuous symmetries in Section 3. Section 4

employs the symmetries calculated in Section 3 to find exact solutions of the system of nonlinear ordinary differential equations derived in Section 2. This results in new families of iso-spectral beams.

2 Iso-spectral deformation through factorization

The main idea is to factorize the canonical beam operator as the product of a linear second-order differential operator and its adjoint, then swap the order of the operators to obtain iso-spectral operators. This idea was already used for vibrating rods by Pöchel and Trubowitz [9]. Ghanbari [10] attempted the factorization method on vibrating beams but failed to solve the resulting system of nonlinear ordinary differential equations.

Let us define the beam operator as the differential operator

$$\mathfrak{L} = \frac{d^4}{dz^4} + \frac{d}{dz} \left(A(z) \frac{d}{dz} \right) + B(z). \quad (10)$$

Factorize the differential operator \mathfrak{L} as

$$\mathfrak{L} = \mathfrak{R}^* \mathfrak{R}, \quad (11)$$

where

$$\mathfrak{R} = \frac{d^2}{dz^2} + r(z) \frac{d}{dz} + s(z), \quad \mathfrak{R}^* = \frac{d^2}{dz^2} - \frac{d}{dz} [r(z) \cdot] + s(z), \quad (12)$$

r and s are smooth functions of their argument, and \mathfrak{R}^* is the adjoint of \mathfrak{R} . Simple reckoning shows that the factorization given in Eq. (11) constraints the functions r and s to satisfy the following system of nonlinear ordinary differential equations [10].

$$r' - r^2 + 2s = A(z) \quad (13)$$

$$s'' - (rs)' + s^2 = B(z), \quad (14)$$

where the prime stands for differentiation with respect to z .

Reversing the operators in the factorization (11), we obtained the operator [10]

$$\hat{\mathfrak{L}} = \mathfrak{R} \mathfrak{R}^* = \frac{d^4}{dz^4} + \frac{d}{dz} \left(\hat{A}(z) \frac{d}{dz} \right) + \hat{B}(z), \quad (15)$$

where

$$\hat{A}(z) = 2s - 3r' - r^2, \quad (16)$$

$$\hat{B}(z) = s^2 + s'' - r''' - rr'' + rs' - sr'. \quad (17)$$

Define an *eigenpair* of \mathfrak{L} as being a pair (ω, U) , where U is not the zero function, such that $\mathfrak{L}U = \omega^2 U$. It can be easily verified that if (ω, U) (resp. $(\hat{\omega}, \hat{U})$) is an eigenpair of \mathfrak{L} (resp. $\hat{\mathfrak{L}}$), then $(\omega, \Re U)$ (resp. $(\hat{\omega}, \Re^* \hat{U})$) is an eigenpair of $\hat{\mathfrak{L}}$ (resp. \mathfrak{L}). Thus, loosely (we have not considered the boundary conditions), \mathfrak{L} and $\hat{\mathfrak{L}}$ are iso-spectral. This motivates the use of factorization in the search for iso-spectral beam.

In order to characterize iso-spectral beams obtained via factorization, one needs to solve Eqs (13)-(14) for r and s . Solving Eq. (13) for s , and substituting into Eq. (14) yield the single nonlinear ordinary differential equation for r

$$r^{(3)} - 3rr'' - \frac{7}{2}r'^2 + 2(2r^2 + A)r' - A'' - r^2A - \frac{A^2}{2} - \frac{r^4}{2} + rA' + 2B = 0. \quad (18)$$

In the next two sections, we shall use Lie symmetry analysis to solve Eq. (18). We shall refer to Eq. (18) as the *principal equation*.

3 Lie symmetry analysis of the principal equation

A symmetry of a differential equation is an invertible transformation of the dependents and dependents variables that leaves the equation unchanged. The main attractive property of symmetries is that they transform solutions into solutions, and better, they provide a systematic route to integration.

Historically, the Norwegian mathematician Sophus Lie (1842-1899) realized that determining all the symmetries of a given equation is a formidable task. However, if we restrict ourself to symmetries that depends continuously on a small parameter and that forms a group (called an *infinitesimal group* in Lie's terminology or Lie group in modern terminology), it is possible to use the machinery of calculus to determine such symmetries in an algorithmic fashion. Strikingly, Sophus Lie discovered that symmetries are road maps to integration.

Here, we are concerned with the determination of the Lie symmetries of Eq. (18). For more details about Lie's symmetry theory, the reader is referred to the books [11,12,13].

A vector field

$$X = \xi(z, r) \frac{\partial}{\partial z} + \eta(z, r) \frac{\partial}{\partial r}, \quad (19)$$

is a point symmetry of Eq. (18) if

$$X^{[3]} \left[r^{(3)} - 3rr'' - \frac{7}{2}r'^2 + 2(2r^2 + A)r' - A'' - r^2A - \frac{A^2}{2} - \frac{r^4}{2} + rA' + 2B \right] \Big|_{\text{Eq. (18)}} = 0. \quad (20)$$

In Eq. (20), the notation $|_{\text{Eq. (18)}}$ means that after expanding the left hand side, Eq. (18) has to be used to eliminate $r^{(3)}$. The operator $X^{[3]}$ is the third prolongation of X defined recursively by

$$X^{[k+1]} = X^{[k]} + \eta^{[k]} \frac{\partial}{\partial r^{(k)}}, \quad X^{[0]} = X, \quad (21)$$

$$\eta^{[k]} = D \left(\eta^{[k-1]} \right) + r^{(k)} D(\xi), \quad \eta^{[0]} = \eta, \quad (22)$$

$$D = \frac{\partial}{\partial z} + r' \frac{\partial}{\partial r} + r'' \frac{\partial}{\partial r'} + \dots. \quad (23)$$

Since the symmetry coefficients ξ and η are independent r' and r'' , Eq. (20) is polynomial in these derivatives. Thus we may set coefficients of the monomials $(r')^m (r'')^n$ to zero in Eq. (20). It results an over-determined system of linear partial differential equations for ξ and η that simplifies to the following equations.

$$\xi = a(z), \quad \eta = -a'(z)r - 2a''(z), \quad (24)$$

$$aA' + 2a'A + 5a^{(3)} = 0, \quad (25)$$

$$B' + \frac{4a'}{a}B = \frac{a'}{a}A^2 + \frac{AA'}{2} + \frac{a''}{a'}A' + \frac{2a'}{a}A'' + \frac{2a^{(3)}}{a}A + \frac{A^{(3)}}{2} + \frac{a^{(5)}}{a} \quad (26)$$

$$3a'A' + 2a''A + aA'' + 5a^{(4)} = 0, \quad (27)$$

where a is an arbitrary smooth function of z . Simple calculations show that

$$\frac{d}{dz}(\text{Eq. (25)}) \equiv \text{Eq. (27)}.$$

Thus Eq. (27) is a mere differential consequence of Eq. (25). In order to solve the remaining equations, we consider the following cases.

Case I: $A = 0$ and $B = 0$. In this case, we find that $a = k_1 z^2 + k_2 z + k_3$, where k_1 , k_2 , and k_3 are arbitrary constants. The symmetry Lie algebra is spanned by the operators

$$X_1 = \frac{\partial}{\partial z}, \quad X_2 = z \frac{\partial}{\partial z} - r \frac{\partial}{\partial r}, \quad X_3 = z^2 \frac{\partial}{\partial z} - 2(rz + 2) \frac{\partial}{\partial r}. \quad (28)$$

The Lie bracket of these operators are:

$$[X_1, X_2] = X_1, \quad [X_1, X_3] = 2X_2, \quad [X_2, X_3] = X_3.$$

The symmetry Lie algebra is non-solvable since its derived algebra of any order is nontrivial. To be more precise, if $\mathfrak{S} = \langle X_1, X_2, X_3 \rangle$, then $\mathfrak{S}^{(k)} = \mathfrak{S}$. In fact, $\mathfrak{S} \cong sl(2, \mathbb{R})$ [14].

Case II: $A \neq 0$ or $B \neq 0$. Solving Eqs. (25)-(26), we obtain after some calculations

$$A = \frac{5a'^2 - 10aa'' + 2C_1}{2a^2}, \quad (29)$$

$$B = \frac{81a'^4 + 12a'^2(3C_1 - 17aa'') + 72a^2a'a^{(3)}}{16a^4} + \frac{4(C_1^2 + 4C_2 - 6C_1aa'' + 21a^2a''^2 - 6a^2a^{(4)})}{16a^4}, \quad (30)$$

where C_1 and C_2 are arbitrary integration constants. The symmetry Lie algebra is spanned by the single operator

$$\Gamma = a \frac{\partial}{\partial z} - (a'r + 2a'') \frac{\partial}{\partial r}. \quad (31)$$

4 Solutions of the principal equation and iso-spectral beams

Here we exploit the symmetry structure of the principal equation (18) to find its solutions.

4.1 A and B are given by Eqs. (29)-(30)

In this case the symmetry Lie algebra is one-dimensional. The best we can do using Lie's integration technique is to reduce the order of the equation by one. This is accomplished by introducing new dependent and independent variables as independent solutions (known as *basis of first order differential invariants*) of

$$\Gamma^{[1]}I = 0. \quad (32)$$

A fundamental set of solutions of Eq. (32) are obtained by solving the system of ordinary differential equations

$$\frac{dz}{a(z)} = \frac{dr}{-a'r - 2a''} = \frac{dr'}{-2a'r' - a''r - 2a^{(3)}}. \quad (33)$$

The solutions of Eq. (33) are

$$ar + 2a' = q_1, \quad a^2r' + aa'r + 2aa'' = q_2, \quad (34)$$

where q_1 and q_2 are arbitrary constants of integration. Thus we introduce the new variables

$$u = ar + 2a', \quad v = a^2 r' + aa' r + 2aa'' . \quad (35)$$

In terms of the new variables, Eq. (18) reads

$$2v^2 \frac{d^2 v}{du^2} = 6uv \frac{dv}{du} - 2v \left(\frac{dv}{du} \right)^2 + u^4 - 8u^2 v + 7v^2 + 2C_1 u^2 - 4C_1 v - 4C_2 . \quad (36)$$

Equation (36) is devoid of Lie symmetries so that we may not further integrate Eq. (36) by Lie's method. Assume from now on that $C_1 = 0 = C_2$, and look for a solution of Eq. (36) in the form

$$v = ku^2, \quad (37)$$

where k is a constant to be determined. Substituting the ansatz (37) into Eq. (36) yields

$$12k^3 - 19k^2 + 8k - 1 = 0. \quad (38)$$

Solving Eq. (38), we obtain

$$k \in \left\{ \frac{1}{4}, \frac{1}{3}, 1 \right\}. \quad (39)$$

Equation (37) may be written as

$$\frac{d}{dz} \left(r + 2\frac{a'}{a} \right) + \frac{a'}{a} \left(r + 2\frac{a'}{a} \right) = k \left(r + 2\frac{a'}{a} \right)^2 . \quad (40)$$

Whence the natural change of variable

$$w = r + 2\frac{a'}{a} . \quad (41)$$

In the new variable, Eq. (40) becomes Bernoulli's equation

$$w' + \frac{a'}{a} w = kw^2. \quad (42)$$

The classical technique for integrating Bernoulli's equation leads to

$$w = \frac{1}{C a - k a \int_0^z a^{-1} dt} , \quad (43)$$

where C is an arbitrary constant of integration. Hence

$$r = \frac{1}{C a - k a \int_0^z a^{-1} dt} - \frac{2a'}{a}, \quad k \in \left\{ \frac{1}{4}, \frac{1}{3}, 1 \right\}. \quad (44)$$

To summarize, we have established the following result.

Theorem 1 *The operator \mathfrak{L} , where*

$$A = \frac{5a'^2 - 10aa''}{2a^2}, \quad (45)$$

$$B = \frac{81a'^4 - 204aa'^2a'' + 72a^2a'a^{(3)} + 84a^2a''^2 - 24a^2a^{(4)}}{16a^4}, \quad (46)$$

is iso-spectral with the operator $\hat{\mathfrak{L}}$, where \hat{A} and \hat{B} are defined by Eq. (16)-(17), r is given by Eq. (44) and $s = (A + r^2 - r')/2$.

4.2 Case of the standard unit-coefficient beam ($A = 0$ and $B = 0$)

According to the symmetry analysis performed in section 3, this is the most symmetric instance of the principal equation. It turns out that, in this particular case, the principal equation can be invertibly mapped to a Chazy equation[15,16,17]. Indeed, Chazy's equation

$$y_{xxx} = 2yy_{xx} - 3y_x^2 + \alpha(6y_x - y^2)^2, \quad \alpha = \text{const.}, \quad (47)$$

admits a 3D symmetry Lie algebra spanned by the operators [18]

$$Y_1 = \frac{\partial}{\partial x}, \quad Y_2 = x \frac{\partial}{\partial x} - y \frac{\partial}{\partial y}, \quad Y_3 = x^2 \frac{\partial}{\partial x} - (2xy + 6) \frac{\partial}{\partial y}. \quad (48)$$

Simple calculations show that $\langle X_1, X_2, X_3 \rangle$ and $\langle Y_1, Y_2, Y_3 \rangle$ are two equivalent (up to an invertible transformation) representations of $sl(2, \mathbb{R})$. Indeed the transformation

$$r = y, \quad z = \frac{2}{3}x, \quad (49)$$

maps $\langle X_1, X_2, X_3 \rangle$ into $\langle Y_1, Y_2, Y_3 \rangle$. The same transformation maps the principal equation into Chazy's equation

$$y_{xxx} = 2yy_{xx} - 3y_x^2 + \frac{4}{27}(6y_x - y^2)^2. \quad (50)$$

The integrability of Eq. (47) was established by Chazy [17]. The justification of the integrability of Eq. (47) from a Lie symmetry standpoint is due to Clarkson and Olver [18].

The integration of Eq.(47) relies on the following theorem.

Theorem 2 (Chazy [17]) *Assume that φ and ψ are two arbitrary linearly independent solutions of the hypergeometric equation*

$$t(1-t)\frac{d^2\chi}{dt^2} + \left(\frac{1}{2} - \frac{7}{6}t\right)\frac{d\chi}{dt} - \sigma\chi = 0, \quad \sigma = \frac{1}{144(1-9\alpha)}. \quad (51)$$

Then the general solution of Eq.(47) is given in parametric form by

$$x = \frac{\varphi(t)}{\psi(t)}, \quad y = \frac{6}{\psi(t)} \frac{d\psi}{dx}. \quad (52)$$

Remark. The case $\alpha = 1/9$ was treated by Clarkson and Olver [18]. In this degenerate case, the solution of Eq. (47) is still given by Eq. (52), where this time, φ and ψ are two arbitrary linearly independent solutions of the Airy equation $\ddot{\chi} + (ct/2)\chi = 0$, and c is an arbitrary constant.

For Eq. (50), $\sigma = -1/48$ and we may choose φ and ψ as [19]

$$\varphi = k_1 {}_2F_1\left(\frac{1}{4}, -\frac{1}{12}, \frac{2}{3}; 1-t\right) - 4^{-2/3}k_2(1-t)^{1/3} {}_2F_1\left(\frac{1}{4}, \frac{7}{12}, \frac{4}{3}; 1-t\right) \quad (53)$$

$$\psi = k_3 {}_2F_1\left(\frac{1}{4}, -\frac{1}{12}, \frac{2}{3}; 1-t\right) - 4^{-2/3}k_4(1-t)^{1/3} {}_2F_1\left(\frac{1}{4}, \frac{7}{12}, \frac{4}{3}; 1-t\right) \quad (54)$$

where ${}_2F_1$ is Gauss hypergeometric function, k_1 to k_4 are arbitrary constants, and the factor $-4^{-2/3}$ is introduced for convenience. Since the formula (52) is invariant under equal scaling of φ and ψ , we may assume without loss of generality that $k_1k_4 - k_2k_3 = -1$, so that Eqs. (53)-(54) involve only three arbitrary constants. Thanks to the exact formulas [20]

$${}_2F_1\left(\frac{1}{4}, -\frac{1}{12}, \frac{2}{3}; \frac{\tau(\tau+4)^3}{4(2\tau-1)^3}\right) = (1-2\tau)^{-1/4} \quad (55)$$

$${}_2F_1\left(\frac{1}{4}, \frac{7}{12}, \frac{4}{3}; \frac{\tau(\tau+4)^3}{4(2\tau-1)^3}\right) = \frac{4(1-2\tau)^{3/4}}{\tau+4}, \quad (56)$$

$$(57)$$

it is natural to introduce the re-parametrization

$$1-t = \frac{\tau(\tau+4)^3}{4(2\tau-1)^3}. \quad (58)$$

Thus φ and ψ reduce to

$$\varphi = (k_1 + k_2\tau^{1/3})(1-2\tau)^{-1/4}, \quad (59)$$

$$\psi = (k_3 + k_4\tau^{1/3})(1-2\tau)^{-1/4}. \quad (60)$$

Therefore the general solution of Eq. (50) is given in parametric form by the equations

$$x = \frac{k_1 + k_2\tau^{1/3}}{k_3 + k_4\tau^{1/3}}, \quad y = \frac{3(k_3 + k_4\tau^{1/3})(3k_3\tau^{2/3} + k_4(2 - \tau))}{1 - 2\tau}. \quad (61)$$

Solving Eq. (61a) for τ and substituting into Eq. (61b) yield

$$y = \frac{3[3k_3(k_3x - k_1)^2(k_4x - k_2) + 2k_4(k_4x - k_2)^3 + k_4(k_3x - k_1)^3]}{(k_4x - k_2)[2(k_1 - k_3x)^3 - (k_4x - k_2)^3]}. \quad (62)$$

Finally, we obtain

$$r = \frac{6[3k_3(3k_3z - 2k_1)^2(3k_4z - 2k_2) + 2k_4(3k_4z - 2k_2)^3 + k_4(3k_3z - 2k_1)^3]}{(3k_4z - 2k_2)[2(2k_1 - 3k_3z)^3 - (3k_4z - 2k_2)^3]} \quad (63)$$

$$s = (r^2 - r')/2, \quad (64)$$

$$k_1k_4 - k_2k_3 = -1. \quad (65)$$

Thus we have established the following result.

Theorem 3 *The unit beam operator $\frac{d^4}{dz^4}$ is iso-spectral with the operator $\hat{\mathcal{L}}$ with*

$$\hat{A} = -5r', \quad (66)$$

$$\hat{B} = -\frac{1}{4}(6r^{(3)} + 2rr'' - r^4 - 7r'^2), \quad (67)$$

where r is defined by Eq. (63).

5 Conclusion

The factorization of the canonical beam operator as the product of a second-order linear differential operator and its adjoint results in a system of nonlinear ordinary differential equations. We uncoupled the system of ordinary differential equations and solved the resulting nonlinear ordinary differential equation (*viz.* the principal equation) using Lie's method. The symmetry analysis reveals that the principal equation admits either a one-dimensional or three-dimensional symmetry Lie algebra. When the principal equation admits a one-dimensional Lie algebra, its order can be reduced by one at most. In this case we obtained a one-parameter family of solutions (see Eq. (44)). For the maximally symmetric case, we proved that the principal equation can be mapped to a Chazy equation. The latter is solved in closed form and the

general solution (see Eq. (63)) of the principal equation is obtained. By reversing the order of the factorization of the canonical beam operator, numerous nontrivial iso-spectral families are obtained.

It is opportune to mention that Nucci [21] calculated the Lie point symmetries of the system (13)-(14) without showing how these symmetries are used. The truth is, for systems of ordinary differential equations, successive reduction of order using symmetries can be ambiguous: there are in general several possibilities for choosing the new variables as invariants [22,23]. We avoided this problem by first uncoupling the system to obtain a single nonlinear ordinary differential equation on which we applied Lie symmetry analysis.

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